Indian Statistical Institute

Final Examination 2013-2014

M.Math First Year

Linear Algebra

Time : 3 Hours Date : 06.11.2013 Maximum Marks : 100 Instructor : Jaydeb Sarkar

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(i) Answer all questions. (ii) All vector spaces are finite dimensional.

Q1. (10 marks) Let V be a vector space over $\mathbb{Q}$. Let $\dim V = 3$ and $T \in \mathcal{L}(V)$. Show that $T^8 = I_V$ if and only if $T^4 = I_V$.

Q2. (10 marks) Let V be an inner product space and $T \in \mathcal{L}(V)$. Prove that T is an isometry if and only if $|T|$ is an isometry.

Q3. (15 marks) State and prove the polar decomposition theorem.

Q4. (10 marks) Let V be an inner product space. Let $T \in \mathcal{L}(V)$ and $c > 0$. Prove that $cI_V + T^*T$ is invertible.

Q5. (10 marks) Let V be a vector and S be a subspace of V and $\dim V = 5$ and $\dim S = 3$. Let $P_S \in \mathcal{L}(V)$ be the orthogonal projection of V onto S. Compute the characteristic polynomial and the minimal polynomial of P_S.

Q6. (10 + 5 marks) Let V be an inner product space and $T \in \mathcal{L}(V)$.

(i) Prove that $\dim[\ker T] = \dim[\ker T^*]$.

(ii) True or false (justify your answer): $\overline{\sigma(T)} = \sigma(T^*)$.

Q7. (10 marks) Let $A \in M_5(\mathbb{C})$ with characteristic polynomial $d_A(z) = (z-2)^3(z+7)^2$ and minimal polynomial $m_A(z) = (z-2)^2(z+7)$. Find the Jordan canonical form for A.

Q8. (15 marks) Let $T \in \mathcal{L}(V)$ and m be an integer with $1 \leq m < \dim V$. Suppose that $T(W) \subseteq W$ for all subspaces $W \subseteq V$ with $\dim W = m$. Prove that $T = cI_V$ for some scalar c.

Q9. (5+10 marks) Let V be a vector space and $\{P_i\}_{i=1}^k \subseteq \mathcal{L}(V)$ such that $\sum_{i=1}^k P_i = I_V$ and $P_iP_j = 0$ for all $i \neq j$. Prove that

(i) P_i is a projection, $1 \leq i \leq k$.

(ii) Let $c_1, \ldots, c_k$ be distinct scalars and $T = \sum_{i=1}^k c_iP_i$. Prove that P_i commutes with T and $P_i = p_i(T)$ for some $p_i \in F[x]$, $1 \leq i \leq k$.